

# The energy-integral method : application to one-phase hyperbolic Stefan problems

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**Abstract**—This paper presents an approximate mathematical technique utilizing the *energy-integral method* for solving the planar solidification problem of a pure liquid metal occupying the infinite half-space. Assuming a time-dependent relaxation model for the energy flux results in a hyperbolic differential equation for the thermal field which is solved under suitable conditions of both local thermodynamic equilibrium and thermal dynamical compatibility on the interface displacement. In fact, analytical expressions are derived when (a) surface temperature is prescribed; or (b) heat flux at the surface boundary is given. Comparisons of these expressions with corresponding results pertinent to parabolic Stefan problems are made; and finally all the solutions are presented in graphical form.

## 1. INTRODUCTION

IN GENERAL, a valid description of a physical phenomenon usually leads to a non-linear problem. In the field of heat transfer, and more particularly heat conduction, the transient heat conduction equation becomes nonlinear if the temperature variations are large or the thermal properties vary rapidly with temperature. On the other hand, when heat transfer takes place at high temperature levels the effects of thermal radiation or a change of phase may occur, and, as a consequence, the boundary conditions become nonlinear. Therefore, a transient heat conduction problem becomes nonlinear due to the nonlinearity of either the differential equation or the boundary conditions or both. The difficulty in the analysis of non-linear problems centers on the fact that no general analytic theory is yet available for the solution of non-linear partial differential equations; each problem should be tackled individually and the principle of superposition not being applicable, the analytic solutions of non-linear problems are often *ad hoc* and approximate. An elegant mathematical technique by which approximate analytic solutions to non-linear transient heat transfer problems can be obtained is the *energy-integral method*. For the application of this method, non-linear problems need not be linearized beforehand.

A rather important class of transient non-linear heat transfer problems are those involving a liquid-solid phase change and which are usually known as Stefan problems. These problems are characterized by an undetermined moving boundary which separates the distinct phases, and which must be determined as part of the solution of the problem. In particular, assuming a time-dependent relaxation model for the energy flux instead of the classical Fourier law gives rise to a hyperbolic differential equation for the thermal field, and, as a consequence to a hyperbolic

Stefan problem. In recent years, many investigators have explored the effect of non-Fourier conduction accompanied with a phase change in transient heat transfer processes. Using similarity arguments, Saad and Didlake [1] solved a one-dimensional melting problem of a semi-infinite solid based on the non-Fourier heat conduction theory. DeSocio and Gualtieri [2] have utilized potential theoretic arguments to obtain an approximate solution for a one-dimensional, two-phase Stefan problem. The authors of ref. [3] have proposed an explicit solution for a one-phase melting problem described along the infinite half-line with a straight line as an interface front, other than a characteristic. Under certain conditions on the interface, Showalter and Walkington [4] have demonstrated the well-posedness of a hyperbolic Stefan problem in the weak sense.

Although the energy-integral method is applicable to all heat transfer problems, it finds its greatest applicability in Stefan problems or transient problems which involves non-linear boundary conditions. In fact, Goodman [5] has developed a method which utilizes the energy-integral and applies it to parabolic Stefan problems. The objective of this paper, however, is to extend the application of Goodman's method to hyperbolic Stefan problems. It will be seen that the equation for determining the location of the interface boundary reduces to an ordinary differential equation the solution of which can frequently be expressed in closed analytical form.

## 2. A RELAXATION MODEL FOR THE ENERGY FLUX

In classical heat transfer phenomena, the constitutive equation governing the heat flow is given by the empirical Fourier law, which relates the heat flow flux with temperature gradient. In particular, the one-

## NOMENCLATURE

$A, C$	constants	$\Delta t$	time mesh
$a(t), b(t), c(t)$	arbitrary time-dependent coefficients, equation (44)	$u(x, t)$	temperature difference
$b$	initial solidification speed	$x$	spatial variable.
$C_p$	specific heat of solid at constant pressure		
$g(x, t)$	heat generation function	Subscript	
$f(t), h(t), r(t), v(t)$	see equation (30)	0	surface value (at $x = 0$ ).
$F(t)$	integral of $f(t)$ , equation (36)	Greek symbols	
$\tilde{v}(t, \tau)$	singular perturbation approximation of $v(t)$	$\alpha$	thermal diffusivity
$K$	thermal conductivity	$\beta$	volumetric latent heat of fusion
$L$	latent heat of fusion	$\gamma$	speed of heat wave propagation
$M$	constant, equation (47b)	$\zeta$	dimensionless time
$N$	constant, equation (27b)	$\eta(t), \nu(t)$	given functions of time, equations (52) and (56)
$q(x, t)$	heat flux	$\Theta$	energy integral
$Q(t), R(t)$	see equations (48) and (58), respectively	$\lambda$	polynomial coefficients, equations (25a) and (50)
$s(t)$	solidification front position	$\mu(t), \mu^*(t)$	specified functions of time, equations (25b) and (59b)
$\bar{s}(\bar{t})$	dimensionless solidification front position	$\xi$	dummy variable
$\tilde{s}(t, \tau)$	approximate solidification front position	$\rho$	density of solid
$Ste$	Stefan number, equation (27b)	$\sigma$	polynomial coefficients, equation (59a)
$T(x, t)$	temperature distribution	$\tau$	relaxation time
$T_f$	fusion temperature	$\phi(t), \psi(\zeta), cp$	components of $\tilde{v}(t, \tau)$ , equation (32).
$t$	time		
$\bar{t}$	dimensionless time		

dimensional Fourier law for a homogeneous isotropic material reads

$$q(x, t) = -KT_x(x, t) \quad (1)$$

where  $q$  is the heat flow flux and  $T$  the temperature distribution within the body.

On the other hand, the energy-balance differential equation for a homogeneous, isotropic material with energy source within the body (or the First Law of Thermodynamics) is

$$-q_x(x, t) + g(x, t) = \rho C_p T_t(x, t) \quad (2)$$

where  $\rho$  is the density of the material,  $C_p$  the specific heat at constant pressure, and  $g$  the volumetric energy source which may be a function of time and/or position. Here and throughout this paper, the thermal characteristics are assumed to be constant.

Combining equations (1) and (2), gives the usual heat conduction equation for a homogeneous isotropic material with a heat source or sink within the body as

$$KT_{xx}(x, t) + g(x, t) = \rho C_p T_t(x, t). \quad (3)$$

This partial differential equation is of the parabolic type and thus has several serious shortcomings, the most prominent one being that a thermal disturbance at any point in a body is instantaneously felt at every point throughout the body, that is, the speed of heat

propagation for Fourier heat conductors is infinite. However, since thermal energy is carried by molecular motion which propagates at a finite speed, one generally concludes that Fourier's law is a low order approximation to a more exact constitutive relation. This has prompted a considerable interest in a higher order approximation for the heat flux model.

In fact, as attempts to circumvent the non-physical phenomenon of an infinite speed of heat propagation, a number of researchers have proposed as a substitute for equation (1) a time-dependent relaxation model for the heat flux law

$$\tau q_t + q = -KT_x \quad (4)$$

as a means of placing an upper bound on the propagation speed of thermal disturbances. Here the constant  $\tau > 0$  is some thermal relaxation or start-up time for the commencement of heat flow after applying a temperature gradient. For many solids,  $\tau$  is of the order of  $10^{-12}$ – $10^{-14}$  s [6]. The use of equation (4) appears to be particularly important in certain transient heat transfer applications with large variations in temperature or large gradients of temperature. Furthermore, with the advent of laser penetration and welding, explosive bonding, fast flux nuclear reactors, and electrical discharge machining, short time, high heat flux melting situations are becoming more prevalent. For various studies on the conduction of heat

based on equation (4), the reader is referred to refs. [7-18].

As pointed out in refs. [1, 19], the non-Fourier effect is generally important only at very early time in transient heat transfer process and decays quickly so that the classical Fourier equation becomes accurate a short time after the initial transient. However, such an effect can be important even at a long time after the initial transient if the thermal disturbance is oscillatory with the period of oscillation of the same order of magnitude as the thermal relaxation time [19].

Equation (4) when substituted into the energy equation (2) yields the relaxation model for the heat flow

$$KT_{xx} + [g + \tau g_t] = \rho C_p [\tau T_{tt} + T_t] \quad (5)$$

Equation (5) is a dissipative wave heat conduction equation with a finite speed of propagation,  $\gamma = \sqrt{(K/\rho C_p \tau)}$ . For infinite propagation speed (i.e.  $\gamma \rightarrow \infty$  or  $\tau \rightarrow 0$ ), equation (5) reduced to its classical heat diffusion version (3). Furthermore, the use of equation (5) removes the peculiarity of an infinite temperature gradient at the boundary as time approaches zero. Finally, note that the temperatures predicted by equations (3) and (5) differ only in non-steady state conditions, since the relaxation model reduces to the Fourier model under steady-state conditions, even when  $\tau \neq 0$ .

### 3. STATEMENT OF THE PROBLEM

Consider the planar solidification, using a certain mechanism, of a pure liquid metal occupying the infinite half-space  $0 \leq x < \infty$ . At time  $t = 0$ , the surface  $x = 0$  is cooled and subsequently maintained at a given time-dependent temperature  $T_0(t)$  ( $T_0(t) < T_f$ ). Immediately the liquid along  $x = 0$  solidifies and subsequently a solidification front moves progressively through the liquid such that behind the front the material is in its solid phase while ahead of the front the metal is in its liquid phase at  $T_f$

$$T(x, t) = T_f, \quad x \geq s(t), \quad t > 0. \quad (6)$$

The problem in the solid phase is then to find at any later time  $t > 0$  a pair of functions  $(s(t), T(x, t))$  satisfying the following conditions:

$$\alpha T_{xx} = \tau T_{tt} + T_t, \quad 0 < x < s(t), \quad t > 0 \quad (7)$$

$$T(0, t) = T_0(t), \quad t > 0 \quad (8)$$

$$T(s(t), t) = T_f, \quad t > 0 \quad (9)$$

$$q(s(t), t) = -\beta \dot{s}(t), \quad t > 0; \quad s(0) = 0 \quad (10)$$

where  $\alpha = K/\rho C_p$  is the heat diffusivity of the metal and the positive constant  $\beta$  is  $\rho L$ ,  $L$  being the latent heat of fusion. Note that equations (9) and (10) express the conditions of local thermodynamic equilibrium and of thermal dynamical compatibility at the solidification front  $x = s(t)$ , respectively.

Condition (10) is not suitable in its present form. However, following ref. [3] it can be rewritten in terms

of the temperature gradient. In fact, differentiating equations (9) and (10) along the solidification front and combining the results with equation (4), one obtains

$$T_x(s(t), t)[\gamma^2 - \dot{s}^2(t)] = A[\ddot{s}(t) + (1/\tau)\dot{s}(t)], \quad t > 0 \quad (11)$$

where  $A = L/C_p$  and  $\gamma^2 = \alpha/\tau$ .

*Remark 1.* The temperature boundary condition on the surface  $x = 0$  can be replaced by the flux boundary condition

$$q(0, t) = q_0(t), \quad t > 0 \quad (12)$$

with prescribed  $q_0(t) > 0$ .

For purposes of reference, the solidification problem (7)-(9), and (11) will be referred to as Problem I. When condition (8) is replaced by condition (12), the solidification problem is termed as Problem II.

### 4. SOLUTION OF PROBLEM I USING THE ENERGY-INTEGRAL METHOD

Define

$$u(x, t) = T_f - T(x, t). \quad (13)$$

Then Problem I reduces to the problem of finding a pair of functions  $(s(t), u(x, t))$  such that

$$\alpha u_{xx} = \tau u_{tt} + u_t, \quad 0 < x < s(t), \quad t > 0 \quad (14)$$

$$u(0, t) = T_f - T_0(t) \equiv u_0(t), \quad t > 0 \quad (15)$$

$$u(s(t), t) = 0, \quad t > 0 \quad (16)$$

$$u_x(s(t), t)[s^2(t) - \gamma^2] = A[\ddot{s}(t) + (1/\tau)\dot{s}(t)], \quad t > 0. \quad (17)$$

Integrating the governing differential equation (14) with respect to  $x$  over  $0 < x < s(t)$  and employing (16) and (17), one obtains

$$\left( \tau \frac{d^2}{dt^2} + \frac{d}{dt} \right) [\Theta(t) + A s(t)] = -\alpha u_x(0, t) \quad (18a)$$

where

$$\Theta(t) = \int_0^{s(t)} u(x, t) dx \quad (18b)$$

is the total energy of the solidified liquid metal.

Equation (18a) is the *energy-integral equation* for this solidification problem. To solve this equation, a second-degree polynomial approximation for the temperature is assumed in the form

$$u(x, t) = a(t)(x-s) - b(t)(x-s)^2 \quad (19)$$

where  $s \equiv s(t)$ . Two conditions are needed to determine the two unknown coefficients  $a(t)$  and  $b(t)$ . Equation (15) provides one condition, and the second condition is essentially (17); but expressed in a more appropriate form as follows.

Differentiating (16) along the curve  $x = s(t)$  gives

$$u_t(s(t), t) = -u_x(s(t), t)\dot{s}(t) \tag{20}$$

so that

$$u_{xx}(s(t), t) = -u_{xx}(s(t), t)\dot{s}(t). \tag{21}$$

From equation (20), it follows that

$$\frac{du_t(s(t), t)}{dt} = -\frac{du_x(s(t), t)}{dt}\dot{s}(t) - u_x(s(t), t)\ddot{s}(t). \tag{22}$$

Simplifying equation (22) and utilizing equation (21) yields

$$u_{tt}(s(t), t) = u_{xxx}(s(t), t)\dot{s}^2(t) - u_x(s(t), t)\ddot{s}(t). \tag{23}$$

Combining equations (14), (17), (20) and (23) leads to

$$Au_{xx}(s(t), t) = [u_x(s(t), t)]^2. \tag{24}$$

Conditions (15) and (24) constitute a complete set of independent relations for the determination of the two unknown coefficients. The resulting temperature profile in light of equation (13) becomes

$$T(x, t) = T_f - A \left[ \lambda(x-s) + \frac{\lambda^2}{2}(x-s)^2 \right] \tag{25a}$$

where

$$\lambda = \frac{1 - \sqrt{1 + \mu}}{s}, \quad \mu \equiv \mu(t) = \frac{2(T_f - T_0(t))}{A}. \tag{25b}$$

Substituting the temperature (25) into the energy-integral equation (18) and performing the indicated operations result in the following non-linear ordinary differential equation for the determination of the location of the solid-liquid interface  $s(t)$ :

$$\left( \tau \frac{d^2}{dt^2} + \frac{d}{dt} \right) [s(5 + \mu + \sqrt{1 + \mu})] = \frac{6\alpha}{s} [1 + \mu - \sqrt{1 + \mu}] \tag{26a}$$

with

$$s(0) = 0, \quad \dot{s}(0) = b. \tag{26b}$$

In the event that  $T_0(t)$  is independent of time  $t$ , equation (26a) becomes

$$s \left( \tau \frac{d^2 s}{dt^2} + \frac{ds}{dt} \right) = N \tag{27a}$$

where the constant  $N$  is defined by

$$N = 6\alpha \frac{1 + 2Ste - \sqrt{1 + 2Ste}}{5 + 2Ste + \sqrt{1 + 2Ste}} \tag{27b}$$

and  $Ste = (T_f - T_0)/A$  is the Stefan number. If, in addition,  $\tau$  is allowed to approach zero, then the initial-value problem (27) and (26b) reduces to

$$s \frac{ds}{dt} = N; \quad s(0) = 0. \tag{28}$$

The solution of equation (28) is

$$s(t) = \sqrt{2Nt} \tag{29}$$

which is the same Goodman's expression for the location of the solidification front  $s \equiv s(t)$  (see, e.g. p. 337 of ref. [5]).

Now, in the interest of obtaining an analytical expression for the hyperbolic solidification front location, one may introduce

$$v(t) = s(t)h(t), \quad f(t) = r(t)h(t) \tag{30a}$$

$$h(t) = 5 + \mu(t) + \sqrt{1 + \mu(t)},$$

$$r(t) = 6\alpha[1 + \mu(t) - \sqrt{1 + \mu(t)}] \tag{30b}$$

then equations (26) read

$$v(t) \left[ \tau \frac{d^2 v(t)}{dt^2} + \frac{dv(t)}{dt} \right] = f(t); \quad v(0) = 0, \tag{31}$$

$$v'(0) = bh(0).$$

Based on what was mentioned in Section 2, one may assume that  $0 < \tau \ll 1$ . Then according to a standard procedure in singular perturbation theory a formal uniform approximation,  $\tilde{v}(t; \tau)$ , for the solution  $v(t)$  of the initial-value problem (31) can be constructed by writing

$$\tilde{v}(t; \tau) \equiv \tilde{s}(t; \tau)h(t) = \phi(t) + \psi(\zeta) - cp \tag{32}$$

where  $\zeta = t/\tau$  and  $\phi(t)$  and  $\psi(\zeta)$  are functions of their arguments which have to be determined. Here  $cp$  denotes the common part.

Firstly, the function  $\psi(\zeta)$  can be obtained by solving the problem

$$\frac{d^2 \psi}{d\zeta^2} + \frac{d\psi}{d\zeta} = 0; \quad \psi(0) = 0, \quad \psi'(0) = b\tau h(0). \tag{33}$$

Then one has

$$\psi(\zeta) = b\tau h(0)(1 - e^{-\zeta}). \tag{34}$$

On the other hand,  $\phi(t)$  satisfies the differential equation

$$\phi \frac{d\phi}{dt} = f(t) \tag{35}$$

so that

$$\phi(t) = \sqrt{2F(t) + C}; \quad F(t) = \int_0^t f(t_1) dt_1, \tag{36}$$

where the arbitrary constant  $C$  is determined from the matching requirement  $\phi(0) = \psi(\infty)$  yielding  $C = b^2\tau^2 h^2(0)$ . Hence

$$\phi(t) = \sqrt{2F(t) + b^2\tau^2 h^2(0)}. \tag{37}$$

The common part,  $cp$ , consists of those terms which cancel out in the matching. In the present case

$$cp = bh(0). \tag{38}$$

Substituting equations (34), (37) and (38) in equation (32) gives

$$\tilde{s}(t, \tau)h(t) = \sqrt{(2F(t) + b^2\tau^2h^2(0)) - b\tau h(0)e^{-t/\tau}}. \tag{39}$$

Equations (25) and (39) form a complete approximate analytic solution of Problem I.

When  $T_0(t)$  is independent of time, so are  $g(t)$  and  $h(t)$ . In this case, equation (39) reduces to

$$\tilde{s}(t, \tau) = \sqrt{(2Nt + b^2\tau^2) - b\tau e^{-t/\tau}} \tag{40}$$

where  $N$  is given by equation (27b). Note that the approximate expression (40) agrees with the exact one (29) when  $\tau$  approaches zero.

For purposes of comparison, one may confine attention to the case of a constant surface temperature and define a proper set of dimensionless variables as follows :

$$\bar{t} = \frac{t}{\tau}, \quad \bar{s} = \frac{s}{\sqrt{N\tau}}, \quad \frac{d\bar{s}}{d\bar{t}} = \frac{(ds/dt)}{b} \tag{41}$$

where  $b$  is taken as a reference speed, of the order of magnitude of the speed of the solidification front, the dimensionless equation of which is  $\bar{x} = \bar{s}(\bar{t})$ .

Then the system of equations (27a) and (26b) and expression (29) become dimensionless as

$$\bar{s} \left( \frac{d^2\bar{s}}{d\bar{t}^2} + \frac{d\bar{s}}{d\bar{t}} \right) = 1, \quad \bar{s}(0) = 0, \quad \frac{d\bar{s}(0)}{d\bar{t}} = 1 \tag{42}$$

$$\bar{s}(\bar{t}) = \sqrt{(2\bar{t})}. \tag{43}$$

It is of interest to consider the difference,  $D(\bar{t})$ , between the hyperbolic and parabolic solidification depths for a fixed surface temperature. For conciseness, one may write  $D(\bar{t}) \equiv$  hyperbolic  $\bar{s}(\bar{t})$  - parabolic  $\bar{s}(\bar{t})$ . Figure 1 shows such a difference vs the dimensionless time  $\bar{t}$ . The behavior of the difference is

consistent with the predictions of the previous works (see, e.g. refs. [1-4, 7, 19]). In fact, the hyperbolic solidification depth clearly differs significantly from the diffusive one for early values of  $\bar{t}$  and decreases eventually to zero with increasing values of  $t/\tau$ , i.e. with decreasing  $\tau$  for a fixed value of the real time variable  $t$ .

*Remark 2.* Assuming that the surface temperature is slow varying with time and the solution of Problem I is approximated by a cubic polynomial

$$u(x, t) = a(t)(x-s) + b(t)(x-s)^2 + c(t)(x-s)^3. \tag{44}$$

The determination of the coefficients  $a, b$  and  $c$  require three constraints on the temperature distribution  $u(x, t)$ . These constraints are (15), (24) and

$$u_{xx}(0, t) = 0, \quad t > 0 \tag{45}$$

which is obtained by differentiating (15) and using the governing differential equation (14).

Then the approximate temperature profile becomes

$$T(x, t) = T_f - A \left[ \lambda(x-s) + \frac{\lambda^2}{2}(x-s)^2 + \frac{\lambda}{6v}(x-s)^3 \right] \tag{46a}$$

$$\lambda = \frac{v}{s} = \frac{3 - \sqrt{(9 + 6\mu)}}{s}. \tag{46b}$$

The energy-integral equation (18a) reduces to

$$s \left( \tau \frac{d^2s}{dt^2} + \frac{ds}{dt} \right) = M \tag{47a}$$

where  $M$  is given by

$$M = 4\alpha \frac{3 + 6Ste - \sqrt{(9 + 12Ste)}}{13 + 6Ste - \sqrt{(9 + 12Ste)}} \tag{47b}$$

and  $Ste$  is once again the Stefan number defined above. Equation (47a) has the same form as equation

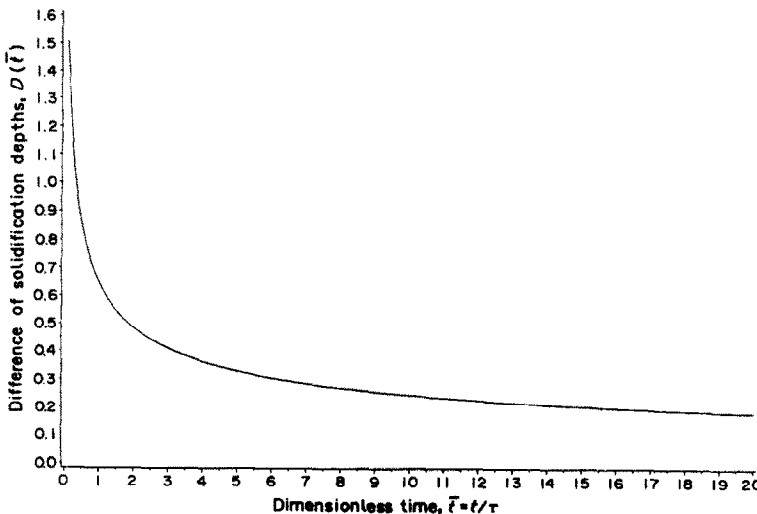


FIG. 1. Difference of solidification depths vs time in dimensionless form for a fixed surface temperature.

(27a) and hence the singular perturbation solution of equation (47a) subject to equation (26b) maintains the same form as equation (40) with  $N$  replaced by  $M$  given above.

**5. SOLUTION OF PROBLEM II USING THE ENERGY-INTEGRAL METHOD**

As mentioned above, Problem II is concerned with the solidification of a pure liquid metal by withdrawal of heat at the surface boundary  $x = 0$ . All the liquid is once again at the solidification (or melting) temperature. Strictly speaking, Problem II is identical with Problem I except that the surface boundary condition (8) is now replaced by condition (12). In mathematical terms, the surface boundary condition at  $x = 0$  may be expressed as

$$KT_x(0, t) = \tau \frac{dq_0(t)}{dt} + q_0(t) \equiv Q(t), \quad t > 0.$$

Then Problem II written in the  $u$ -variable consists of equations (14), (16), (17) and

$$-Ku_x(0, t) = Q(t), \quad t > 0. \tag{48}$$

In the present case, the energy-integral equation (18a) implies that

$$\int_0^{s(t)} u(x, t) dx + As(t) = \frac{\alpha}{K} \int_0^r [1 - e^{-(t-\xi)\tau}] Q(\xi) d\xi \tag{49}$$

where (26b) has been used.

Now, to solve equation (49) for  $s(t)$ , a second-degree polynomial for  $u(x, t)$  is sought in the form of equation (19) where  $a(t)$  and  $b(t)$  are determined according to the two conditions (24) and (48). The resulting temperature profile has the form of equation (25a) where  $\lambda$  is given by

$$\lambda = \frac{1 - \sqrt{1 + 4\mu}}{2s}, \quad \mu \equiv \mu(t) = \frac{Q(t)s(t)}{AK}. \tag{50}$$

Combining such an expression for temperature and equation (49) yields

$$\eta = \frac{\mu}{6} [5 + \mu + \sqrt{1 + 4\mu}] \tag{51a}$$

where

$$\eta = \frac{\alpha Q(t)}{A^2 K^2} \int_0^r [1 - e^{-(t-\xi)\tau}] Q(\xi) d\xi. \tag{51b}$$

The graphical representation of equation (51a) is given in Fig. 2. Note that the resulting curve and the curve for  $\tau \rightarrow 0$  have the same features.

It is evident that the right-hand side of equation (51a) is a monotonically increasing function of  $\mu$ , so that for any given  $\eta$ , there is at most one  $\mu$  which can satisfy (51a). Hence in principle it is possible to uniquely invert equation (51a). With this in mind, one may express  $\eta$  in terms of powers of  $\mu$ , in the form

$$\eta = \mu + \frac{1}{2}\mu^2 - \frac{1}{3}\mu^3 + \frac{2}{3}\mu^4 - \frac{5}{3}\mu^5 + \dots \tag{52}$$

Then it is easily verified that this series is inverted into

$$\mu = \eta - \frac{\eta^2}{2!} + \frac{5}{3!}\eta^3 - \frac{51}{4!}\eta^4 + \frac{795}{5!}\eta^5 - \dots \tag{53}$$

The temperature-time history on the surface boundary  $x = 0$  is given parametrically by equation (51a) and

$$\frac{T_f - T(0, t)}{A} = \frac{1}{4}[-1 + 2\mu + \sqrt{1 + 4\mu}]. \tag{54}$$

These two equations can be cross-plotted to determine  $(T_f - T(0, t))/A$  in terms of  $\eta$ , and the result is shown in Fig. 3, which is the same curve obtained by Goodman [5] for the corresponding diffusive Stefan problem, that is, when  $\tau \rightarrow 0$ .

From equations (50) and (53), it follows that

$$s(t) = \frac{AK}{Q(t)} \left[ \eta - \frac{1}{2!}\eta^2 + \frac{5}{3!}\eta^3 - \frac{51}{4!}\eta^4 + \frac{795}{5!}\eta^5 - \dots \right] \tag{55}$$

where  $\eta \equiv \eta(t)$  is given by equation (51b). If  $\tau$  is allowed to approach zero, then equation (54) reduces to

$$s(t) = \frac{AK}{q_0(t)} \left[ v - \frac{1}{2!}v^2 + \frac{5}{3!}v^3 - \frac{51}{4!}v^4 + \frac{795}{5!}v^5 - \dots \right] \tag{56a}$$

with  $v \equiv v(t)$  defined by

$$v(t) = \frac{\alpha Q(t)}{A^2 K^2} \int_0^r Q(\xi) d\xi. \tag{56b}$$

Equations (25a), (50) and (55) constitute a complete approximate solution of Problem II. If equation (55) is replaced by equation (56), one obtains the solution of the corresponding diffusive problem with

$$\mu = \frac{q_0(t)s(t)}{AK}. \tag{57}$$

*Remark 3.* In approximating the solution of Problem II by a cubic polynomial of the form of equation (44), three conditions are required to determine the coefficients of the polynomial. These conditions are (24), (48) and an additional equation. The extra equation can be obtained by differentiating equation (48) with respect to time and using the governing differential equation to deduce the constraint

$$u_{xxx}(0, t) = -\frac{\tau Q''(t) + Q'(t)}{\alpha} \equiv R(t). \tag{58}$$

In the event that  $Q(t)$  is so slow varying with time that  $R(t)$  vanishes, constraint (58) obviously reduces

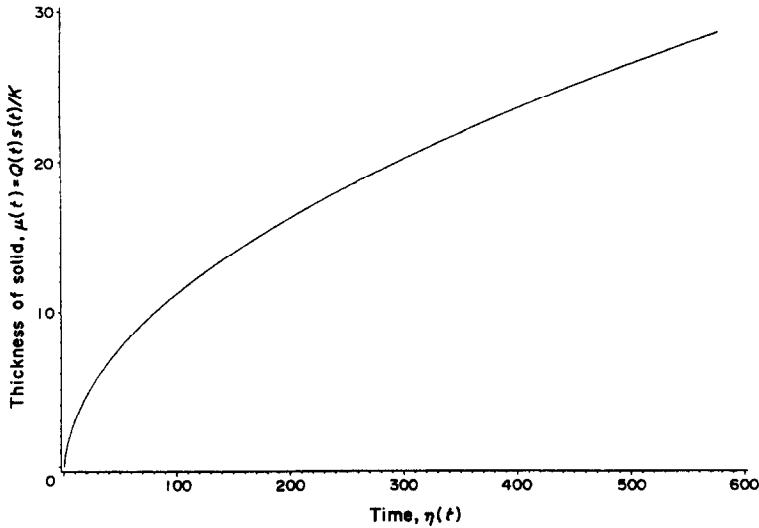


FIG. 2. Thickness of solid vs time for an arbitrary heat flux at the surface boundary; equation (51).

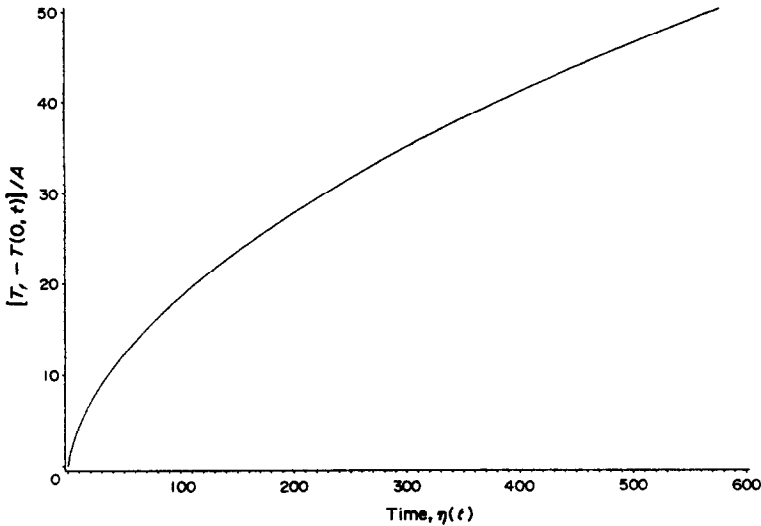


FIG. 3. Temperature-time history on the surface boundary for an arbitrary heat flux at that boundary; equations (54) and (51).

the cubic polynomial (44) to a quadratic expression. Otherwise, combining (24), (44), (48) and (58) results in

$$T(x, t) = T_r - A \left[ \sigma(x-s) + \frac{\sigma^2}{2}(x-s)^2 + \frac{R}{6A}(x-s)^3 \right] \quad (59a)$$

where

$$\sigma = \frac{1 - \sqrt{1 + 4\mu^*}}{2s}; \quad \mu^* = \frac{s}{A} \left( \frac{Q}{K} + \frac{Rs^2}{2} \right). \quad (59b)$$

Inserting equations (59) into equation (49) yields the same form of (51a) except that  $\mu$  and  $\eta$  are now replaced by  $\mu^*$  and

$$\eta^* = \frac{1}{A^2} \left( \frac{Q}{K} + \frac{Rs^2}{2} \right) \times \left( \frac{Rs^4}{24} + \frac{\alpha}{K} \int_0^t [1 - e^{-(t-s)/\tau}] Q(\xi) d\xi \right). \quad (60)$$

Therefore, the analysis after equation (51b) applies in the present case. However, it will not be duplicated here.

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#### METHODE INTEGRALE D'ENERGIE: APPLICATION AUX PROBLEMES HYPERBOLIQUES DE STEFAN MONOPHASIQUES

**Résumé**—On présente une technique mathématique approchée qui utilise la *méthode intégrale d'énergie* pour résoudre le problème de solidification plane d'un métal liquide pur occupant un demi-espace infini. En supposant un modèle de relaxation variable dans le temps, pour le flux d'énergie, le champ thermique est décrit par une équation hyperbolique aux dérivées partielles qui est résolue pour des conditions d'équilibre thermodynamique local et de compatibilité thermique dynamique sur le déplacement de l'interface. De fait des expressions analytiques sont obtenues lorsque: (a) la température de la surface est donnée, ou (b) le flux thermique pariétal est donné. On fait des comparaisons entre ces expressions et celles qui résultent des problèmes paraboliques de Stefan; enfin, les solutions sont présentées sous forme graphique.

#### DIE ENERGIE-INTEGRAL-METHODE: ANWENDUNG AUF EINPHASIGE HYPERBOLISCHE STEFAN-PROBLEME

**Zusammenfassung**—Es wird ein mathematisches Näherungsverfahren vorgestellt, das die Energie-Integral-Methode benutzt, um das ebene Problem der Erstarrung eines flüssigen Metalls zu lösen, welches einen halbinendlichen Raum einnimmt. Die Verwendung eines örtlich-zeitlichen Relaxationsmodells für den Energiestrom führt zu einer hyperbolischen Differentialgleichung für das Temperaturfeld, welche bei geeigneten Bedingungen sowohl für lokales thermodynamisches Gleichgewicht als auch für thermisch-dynamische Verträglichkeit an den sich verschiebenden Grenzflächen gelöst wird. Es werden analytische Ausdrücke hergeleitet, wenn (a) die Oberflächentemperatur oder (b) die Wärmestromdichte an der Oberfläche vorgegeben ist. Diese Ausdrücke werden mit entsprechenden Ergebnissen für parabolische Stefan-Probleme verglichen. Abschließend werden alle Ergebnisse grafisch dargestellt.

#### МЕТОД ИНТЕГРАЛА ЭНЕРГИИ: ПРИМЕНЕНИЕ К ОДНОФАЗНЫМ ГИПЕРБОЛИЧЕСКИМ ЗАДАЧАМ СТЕФАНА

**Аннотация**—Представлен математический способ приближенного решения плоской задачи затвердевания чистого жидкого металла, заполняющего полупространство, основанный на использовании метода интеграла энергии. В результате применения модели нестационарной релаксации для потока энергии получено гиперболическое дифференциальное уравнение для температурного поля, которое решается в условиях, соответствующих как локальному термодинамическому равновесию, так и тепловой динамической совместности при смещении межфазной границы. Получены аналитические выражения в случаях задания (а) температуры поверхности или (б) теплового потока у граничной поверхности. Проведены сравнения данных выражений с соответствующими результатами, полученными для параболической задачи Стефана, а затем все решения представлены в графической форме.